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REVISIT OF IDENTITIES FOR DAEHEE NUMBERS ARISING FROM NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Recently, Kwon-Kim-Seo introduced some interesting identities of Daehee numbers arising from differential equation.
In this paper, we consider the inverse problem for the some identities of Daehee numbers which are derived from differential equations in [6].

1. Introduction

The Bernoulli numbers are defined by the generating function to be

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad \text{(see [1 − 7])}. \tag{1.1}$$

By replacing t by $\log(1+t)$ in (1.1), we get

$$\begin{aligned}\frac{\log(1+t)}{t}&=\sum_{m=0}^{\infty}B_m\frac{1}{m!}\big(\log(1+t)\big)^m\\&=\sum_{m=0}^{\infty}B_m\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\big(\sum_{m=0}^{n}S_1(n,m)B_m\big)\frac{t^n}{n!},\end{aligned} \tag{1.2}$$

where $S_1(n,m)$ is the stirling number of the first kind.

As is well known, Daehee numbers are defined by

$$D_n=\sum_{m=0}^{n}S_1(n,m)B_m,\ (n\geq 0). \tag{1.3}$$

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From (1.2) and (1.3), we note that the generating function of Daehee numbers is given by

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad \text{(see [2])}. \tag{1.4}$$

The Daehee polynomials are also defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \tag{1.5}$$

Thus, by (1.4) and (1.5), we easily get

$$D_n(x) = \sum_{l=0}^{n} \binom{n}{l} (x)_l D_{n-l}, \ (n \geq 0), \tag{1.6}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1)$.
From (1.4), we have

$$\sum_{n=0}^{\infty} D_n \frac{(e^t-1)^n}{n!} = \frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{1.7}$$

It is not difficult to show that

$$\begin{aligned} \sum_{n=0}^{\infty} D_n \frac{1}{n!}(e^t-1)^n &= \sum_{n=0}^{\infty} D_n \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{m} D_n S_2(m,n) \Big) \frac{t^m}{m!}. \end{aligned} \tag{1.8}$$

By (1.7) and (1.8), we get

$$B_m = \sum_{n=0}^{m} D_n S_2(m,n), \ (m \geq 0),$$

where $S_2(m,n)$ is the stirling number of the second kind.
Recently, Kwon-Kim-Seo introduced the following differential equation (see [6]) :

$$\left(\frac{d}{dt}\right)^N \left(\frac{\log(1+t)}{t} \cdot t\right) = \left(\frac{d}{dt}\right)^N \log(1+t) = \sum_{m=0}^{\infty} (m+N) D_{m+N-1} \frac{t^m}{m!}. \tag{1.9}$$

For $r \in \mathbb{N}$, the higher-order Daehee numbers are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r = \sum_{n=0}^{\infty} D_n^{(r)} \frac{t^n}{n!}, \quad \text{(see [2])}.$$

In this paper, with the viewpoint of the inverse problem of (1.9), we derive some new identities of Daehee numbers arising from nonlinear differential equation.

2. Some identities of Daehee numbers

Let

$$F = F(t) = \frac{1}{e^{\log(1+t)} - 1}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{d}{dt}F(t) &= -\left(\frac{1}{e^{\log(1+t)} - 1}\right)^2 \left(\frac{e^{\log(1+t)}}{1+t}\right) \\ &= -\frac{1}{1+t}(F + F^2). \end{aligned} \tag{2.2}$$

From(2.2), we have

$$(1+t)F^{(1)} = -(F + F^2), \tag{2.3}$$

and

$$F^2 = -F - (1+t)F^{(1)}. \tag{2.4}$$

Let us take the derivative on the both sides of (2.4). Then we have

$$2FF^{(1)} = -F^{(1)} - F^{(1)} - (1+t)F^{(2)} = -2F^{(1)} - (1+t)F^{(2)}. \tag{2.5}$$

Thus, by multiply (1+t) on the both sides of (2.5), we get

$$2F(1+t)F^{(1)} = -2F^{(1)}(1+t) - (1+t)^2F^{(2)}, \tag{2.6}$$

and

$$-2F(F + F^2) = -2(1+t)F^{(1)} - (1+t)^2F^{(2)}. \tag{2.7}$$

Thus, by(2.7), we get

$$\begin{aligned} 2F^3 &= -2F^2 + (-1)^2 2(1+t)F^{(1)} + (-1)^2(1+t)^2F^{(2)} \\ &= (-1)^2 2F + (-1)^2 4(1+t)F^{(1)} + (-1)^2(1+t)^2F^{(2)}. \end{aligned} \tag{2.8}$$

Taking derivative on the both sides of (2.8), we have

$$\begin{aligned} 3!F^2F^{(1)} &= (-1)^2 2F^{(1)} + (-1)^2 4F^{(1)} + (-1)^2 4(1+t)F^{(2)} \\ &\quad + 2(-1)^2(1+t)F^{(2)} + (-1)^2(1+t)^2F^{(3)} \\ &= (-1)^2 6F^{(1)} + (-1)^2 6(1+t)F^{(2)} + (-1)^2(1+t)^2F^{(3)}, \end{aligned} \tag{2.9}$$

where $\left(\frac{d}{dt}\right)^N F = \left(\frac{d}{dt}\right)^N F(t) = F^{(N)}$, $(N \in \mathbb{N})$.
Multiply $(1+t)$ on the both sides of (2.9), we get

$$3!F^2(1+t)F^{(1)} = (-1)^2 6(1+t)F^{(1)} + (-1)^2 6(1+t)^2 F^{(2)} + (-1)^2(1+t)^3 F^{(3)}. \tag{2.10}$$

Thus, by (2.3), we get

$$3!F^2(-1)(F+F^2) = (-1)^2 6(1+t)F^{(1)} + (-1)^2 6(1+t)^2 F^{(2)} + (-1)^2(1+t)^3 F^{(3)}. \tag{2.11}$$

By (2.11), we get

$$\begin{aligned} 3!F^4 &= -6F^3 + (-1)^3 6(1+t)F^{(1)} + (-1)^3 6(1+t)^2 F^{(2)} + (-1)^3(1+t)^3 F^{(3)} \\ &= -3\big((-1)^2 2F + (-1)^2 4(1+t)F^{(1)} + (-1)^2(1+t)^2 F^{(2)}\big) \\ &\quad + (-1)^3 6(1+t)F^{(1)} + (-1)^3 6(1+t)^2 F^{(2)} + (-1)^3(1+t)^3 F^{(3)} \\ &= (-1)^3 6F + (-1)^3 18(1+t)F^{(1)} + (-1)^3 9(1+t)^2 F^{(2)} + (-1)^3(1+t)^3 F^{(3)}. \end{aligned} \tag{2.12}$$

Continuing this process, we can set

$$N!F^{N+1} = (-1)^N \sum_{k=0}^{N} a_k(N)(1+t)^k F^{(k)}, \ (N \in \mathbb{N}), \tag{2.13}$$

where

$$F^{N+1} = \underbrace{F \times F \times \cdots \times F}_{N+1-\text{times}}, \ F^{(k)} = \left(\frac{d}{dt}\right)^k F(t).$$

Let us take the derivative on the both sides of (2.13) with respect to t.
Then we have

$$(N+1)!F^N F^{(1)} = (-1)^N \sum_{k=0}^{N} a_k(N)\left(k(1+t)^{k-1}F^{(k)} + (1+t)^k F^{(k+1)}\right). \tag{2.14}$$

Thus, by (2.14), we get

$$\begin{aligned} &(N+1)!F^N(1+t)F^{(1)} \\ &= (-1)^N \sum_{k=0}^{N} a_k(N)\left\{(k(1+t)^k F^{(k)} + (1+t)^{k+1} F^{(k+1)}\right\}. \end{aligned} \tag{2.15}$$

From (2.3) and (2.15), we have

$$\begin{aligned}&(N+1)!F^N(F+F^2)\\&=(-1)^{N+1}\sum_{k=0}^{N}a_k(N)\big(k(1+t)^kF^{(k)}+(1+t)^{k+1}F^{(k+1)}\big).\end{aligned}\tag{2.16}$$

By (2.13) and (2.16), we get

$$\begin{aligned}(N+1)!F^{N+2}&=(-1)^{N+1}\sum_{k=1}^{N}a_k(N)k(1+t)^kF^{(k)}\\&+(-1)^{N+1}\sum_{k=0}^{N}a_k(N)(1+t)^{k+1}F^{(k+1)}-(N+1)N!F^{N+1}\\&=(-1)^{N+1}\sum_{k=1}^{N}a_k(N)k(1+t)^kF^{(k)}\\&+(-1)^{N+1}\sum_{k=1}^{N+1}a_{k-1}(N)(1+t)^kF^{(k)}\\&+(N+1)(-1)^{N+1}\sum_{k=0}^{N}a_k(N)(1+t)^kF^{(k)}\\&=(-1)^{N+1}\sum_{k=1}^{N}(k+N+1)a_k(N)(1+t)^kF^{(k)}\\&+(-1)^{N+1}\sum_{k=1}^{N}a_{k-1}(N)(1+t)^kF^{(k)}\\&+(N+1)(-1)^{N+1}a_0(N)F+(-1)^{N+1}a_N(N)(1+t)^{N+1}F^{(N+1)}\\&=(-1)^{N+1}\sum_{k=1}^{N}\Big\{(k+N+1)a_k(N)+a_{k-1}(N)\Big\}(1+t)^kF^{(k)}\\&+(N+1)(-1)^{N+1}a_0(N)F+(-1)^{N+1}a_N(N)(1+t)^{N+1}F^{(N+1)}.\end{aligned}\tag{2.17}$$

By replacing N by $N+1$ in (2.13), we get

$$(N+1)!F^{N+2}=(-1)^{N+1}\sum_{k=0}^{N+1}a_k(N+1)(1+t)^kF^{(k)}.\tag{2.18}$$

Comparing the coefficients on the both sides of (2.17) and (2.18), we have

$$a_0(N+1) = (N+1)a_0(N), \qquad a_{N+1}(N+1) = a_N(N), \tag{2.19}$$

and

$$(N+1+k)a_k(N) + a_{k-1}(N) = a_k(N+1), \ (1 \le k \le N). \tag{2.20}$$

From (2.4) and (2.13), we have

$$-F - (1+t)F^{(1)} = F^2 = -a_0(1)F - a_1(1)(1+t)F^{(1)}. \tag{2.21}$$

By (2.21), we get

$$a_0(1) = 1 \ \ and \ \ a_1(1) = 1. \tag{2.22}$$

Thus, by (2.19), (2.22), we have

$$\begin{aligned} a_0(N+1) &= (N+1)a_0(N) = (N+1)Na_0(N-1) = \cdots = (N+1)N\cdots 2\cdot a_0(1) \\ &= (N+1)N(N-1)\cdots 2\cdot 1 = (N+1)!, \end{aligned} \tag{2.23}$$

and

$$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1. \tag{2.24}$$

For $1 \le k \le N$, by (2.20), we have

$$\begin{aligned} a_k(N+1) &= (N+1+k)a_k(N) + a_{k-1}(N) \\ &= (N+1+k)\Big\{(N+k)a_k(N-1) + a_{k-1}(N-1)\Big\} + a_{k-1}(N) \\ &= (N+1+k)(N+k)a_k(N-1) + (N+1+k)a_{k-1}(N-1) + a_{k-1}(N) \\ &= (N+1+k)(N+k)\Big\{(N-1+k)a_k(N-2) + a_{k-1}(N-2)\Big\} \\ &\quad + (N+1+k)a_{k-1}(N-1) + a_{k-1}(N) \\ &= (N+1+k)_3 a_k(N-2) + (N+1+k)_2 a_{k-1}(N-2) \\ &\quad + (N+1+k)_1 a_{k-1}(N-1) + a_{k-1}(N) \end{aligned} \tag{2.25}$$

$$
\begin{aligned}
&= (N+1+k)_3 a_k(N-2) + \sum_{n=N-2}^{N} (N+1+k)_{N-n} a_{k-1}(n) \\
&= \cdots \\
&= (N+1+k)_{N-k+1} a_k(k) + \sum_{n_1=k}^{N} (N+1+k)_{N-n_1} a_{k-1}(n_1) \\
&= \sum_{n_1=k-1}^{N} (N+1+k)_{N-n_1} a_{k-1}(n_1).
\end{aligned} \tag{2.26}
$$

From (2.25), we note that

$$
\begin{aligned}
a_k(N+1) &= \sum_{n_1=k-1}^{N} (N+1+k)_{N-n_1} a_{k-1}(n_1) \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} (N+1+k)_{N-n_1} (n_1+k-1)_{n_1-1-n_2} a_{k-2}(n_2) \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \sum_{n_3=k-3}^{n_2-1} (N+1+k)_{N-n_1} (n_1+k-1)_{n_1-1-n_2} \\
&\quad \times (n_2+k-2)_{n_2-1-n_3} a_{k-3}(n_3) \\
&= \cdots \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} (N+1+k)_{N-n_1} (n_1+k-1)_{n_1-1-n_2} \\
&\quad \times \cdots \times (n_{k-1}+1)_{n_{k-1}-1-n_k} a_0(n_k) \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} (N+1+k)_{N-n_1} (n_1+k-1)_{n_1-1-n_2} \\
&\quad \times \cdots \times (n_{k-1}+1)_{n_{k-1}-1-n_k} n_k! \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} \frac{(N+1+k)!}{\prod_{l=1}^{k} (n_l+k-l+2)(n_l+k-l+1)}.
\end{aligned} \tag{2.27}
$$

Therefore, we obtain the following theorem

Theorem 2.1. (**Fundamental identity**)

For $N\in\mathbb{N}$, *we have*

$$N!F^{N+1}=(-1)^N\sum_{k=0}^{N}a_k(N)(1+t)^kF^{(k)},$$

where $a_0(N)=N!$, $a_N(N)=1$, *and*

$$a_k(N)=\sum_{n_1=k-1}^{N-1}\sum_{n_2=k-2}^{n_1-1}\cdots\sum_{n_k=0}^{n_{k-1}-1}\frac{(N+k)!}{\prod_{l=1}^{k}(n_l+k-l+2)(n_l+k-l+1)}.$$

From Theorem 2.1, we note that

$$\begin{aligned} N!\left(\frac{\log(1+t)}{t}\right)^{N+1}&=(-1)^N\sum_{k=0}^{N}a_k(N)(1+t)^k\big(\log(1+t)\big)^{N+1}(-1)^kk!\frac{1}{t^{k+1}}\\ &=(-1)^N\sum_{k=0}^{N}a_k(N)(1+t)^k\big(\log(1+t)\big)^{N-k}(-1)^kk!\left(\frac{\log(1+t)}{t}\right)^{k+1}. \end{aligned}\tag{2.28}$$

Now, we observe that

$$\begin{aligned} &(1+t)^k\big(\log(1+t)\big)^{N-k}(-1)^kk!\left(\frac{\log(1+t)}{t}\right)^{k+1}\\ &=(N-k)!k!(-1)^k\left(\sum_{l_1=0}^{\infty}\binom{k}{l_1}t^{l_1}\right)\left(\sum_{l_2=N-k}^{\infty}S_1(l_2,N-k)\frac{t^{l2}}{l_2!}\right)\left(\frac{\log(1+t)}{t}\right)^{k+1}\\ &=(N-k)!k!(-1)^k\left(\sum_{l_3=N-k}^{\infty}\sum_{l_2=N-k}^{l_3}S_1(l_2,N-k)(k)_{l_3-l_2}\binom{l_3}{l_2}\frac{t^{l_3}}{l_3!}\right)\\ &\quad\times\left(\sum_{l_4=0}^{\infty}D_{l_4}^{(k+1)}\frac{t^{l_4}}{l_4!}\right)\\ &=(N-k)!k!(-1)^k\\ &\quad\times\sum_{n=N-K}^{\infty}\left(\sum_{l_3=N-k}^{n}\sum_{l_2=N-k}^{l_3}\binom{l_3}{l_2}\binom{n}{l_3}S_1(l_2,N-k)(k)_{l_3-l_2}D_{n-l_3}^{(k+1)}\right)\frac{t^n}{n!}. \end{aligned}\tag{2.29}$$

From (2.29), we have

$$
\begin{aligned}
&(-1)^N \sum_{k=0}^{N} a_k(N)(1+t)^k \big(\log(1+t)\big)^{N+1} (-1)^k \frac{k!}{t^{k+1}} \\
&= (-1)^N \sum_{k=0}^{N} a_k(N)(1+t)^k \big(\log(1+t)\big)^{N-k} (-1)^k k! \left(\frac{\log(1+t)}{t}\right)^{k+1} \\
&= (-1)^N \sum_{k=0}^{N} a_k(N)(N-k)!k!(-1)^k \\
&\quad \times \sum_{n=N-k}^{\infty} \Big(\sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, N-k)(k)_{l_3-l_2} D_{n-l_3}^{(k+1)} \Big) \frac{t^n}{n!} \\
&= (-1)^N \sum_{k=0}^{N} a_k(N)(N-k)!k!(-1)^k \\
&\quad \times \Bigg\{ \sum_{n=N-k}^{N} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, N-k)(k)_{l_3-l_2} D_{n-l_3}^{(k+1)} \frac{t^n}{n!} \\
&\qquad + \sum_{n=N+1}^{\infty} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, N-k)(k)_{l_3-l_2} D_{n-l_3}^{(k+1)} \frac{t^n}{n!} \Bigg\} \\
&= \sum_{k=0}^{N} a_{N-k}(N)(N-k)!k!(-1)^k \sum_{n=k}^{N} \sum_{l_3=k}^{n} \sum_{l_2=k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, k)(N-k)_{l_3-l_2} \\
&\quad \times D_{n-l_3}^{(N-k+1)} \frac{t^n}{n!} + \sum_{n=N+1}^{\infty} \Big(\sum_{k=0}^{N} a_k(N)(N-k)!k!(-1)^{N-k} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \binom{l_3}{l_2} \\
&\quad \times \binom{n}{l_3} S_1(l_2, N-k)(k)_{l_3-l_2} D_{n-l_3}^{(k+1)} \Big) \frac{t^n}{n!} \\
&= \sum_{n=0}^{N} \Big(\sum_{k=0}^{n} \sum_{l_3=k}^{n} \sum_{l_2=k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, k)(N-k)_{l_3-l_2} a_{N-k}(N) k!(N-k)!(-1)^k \\
&\quad \times D_{n-l_3}^{(N-k+1)} \Big) \frac{t^n}{n!} + \sum_{n=N+1}^{\infty} \left(\sum_{k=0}^{N} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \binom{l_3}{l_2}\binom{n}{l_3} S_1(l_2, N-k)(k)_{l_3-l_2} \right. \\
&\quad \left. \times a_k(N) D_{n-l_3}^{(k+1)} (N-k)!k!(-1)^{N-k} \right) \frac{t^n}{n!} \\
&= N! \Bigg\{ \sum_{n=0}^{N} \Big(\sum_{k=0}^{n} \sum_{l_3=k}^{n} \sum_{l_2=k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2, k)(N-k)_{l_3-l_2} a_{N-k}(N) D_{n-l_3}^{(N-k+1)} \\
&\quad \times (-1)^k \Big) \frac{t^n}{n!} + \sum_{n=N+1}^{\infty} \Big(\sum_{k=0}^{N} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2, N-k)(k)_{l_3-l_2} \\
&\quad \times a_k(N) D_{n-l_3}^{(k+1)} (-1)^{N-k} \Big) \frac{t^n}{n!} \Bigg\}.
\end{aligned} \tag{2.30}
$$

By (2.29) and (2.30), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_n^{(N+1)} \frac{t^n}{n!} = \Big(\frac{\log(1+t)}{t}\Big)^{N+1}\\
&= \sum_{n=0}^{N} \Big(\sum_{k=0}^{n} \sum_{l_3=k}^{n} \sum_{l_2=k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2,k)(N-k)_{l_3-l_2} a_{N-k}(N) D_{n-l_3}^{(N-k+1)} (-1)^k \Big) \frac{t^n}{n!}\\
&+ \sum_{n=N+1}^{\infty} \Big(\sum_{k=0}^{N} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2,N-k)(k)_{l_3-l_2} a_k(N) D_{n-l_3}^{(k+1)}\\
&\quad \times (-1)^{n-k} \Big) \frac{t^n}{n!}.
\end{aligned} \tag{2.31}$$

Therefore, by (2.31), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, we have*

$$D_n^{(N+1)} = \sum_{k=0}^{n} \sum_{l_3=k}^{n} \sum_{l_2=k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2,k)(N-k)_{l_3-l_2} a_{N-k}(N) D_{n-l_3}^{(N-k+1)} (-1)^k$$

where $0 \le n \le N$. *For* $N+1 \le n$, *we have*

$$D_n^{(N+1)} = \sum_{k=0}^{N} \sum_{l_3=N-k}^{n} \sum_{l_2=N-k}^{l_3} \frac{\binom{l_3}{l_2}\binom{n}{l_3}}{\binom{N}{k}} S_1(l_2,N-k)(k)_{l_3-l_2} a_k(N) D_{n-l_3}^{(k+1)} (-1)^{n-k}.$$

From Theorem 2.1, we have

$$\begin{aligned}
&N!\Big(\frac{\log(1+t)}{t}\Big)^{N+1} = (-1)^N \big(\log(1+t)\big)^{N+1} \sum_{k=0}^{N} a_k(N)(1+t)^k F^{(k)}\\
&= (-1)^N \big(\log(1+t)\big)^{N+1} \sum_{k=0}^{N} a_k(N)(1+t)^k \Big(\frac{d}{dt}\Big)^k \Big(\frac{\log(1+t)}{t} \cdot \frac{1}{\log(1+t)}\Big).
\end{aligned} \tag{2.32}$$

As is well known, the Bernoulli numbers of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}. \tag{2.33}$$

We note that

$$\begin{aligned}
&\Big(\frac{d}{dt}\Big)^k\Big(\frac{\log(1+t)}{t}\cdot\frac{1}{\log(1+t)}\Big)\\
&=\Big(\frac{d}{dt}\Big)^k\Big(\frac{1}{\log(1+t)}+\sum_{m=0}^{\infty}\frac{D_{m+1}}{m+1}\cdot\frac{t^m}{m!}\cdot\frac{t}{\log(1+t)}\Big)\\
&=\Big(\frac{d}{dt}\Big)^k\left\{\frac{1}{\log(1+t)}+\Big(\sum_{m=0}^{\infty}\frac{D_{m+1}}{m+1}\cdot\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}b_l\frac{t^l}{l!}\Big)\right\}\\
&=\Big(\frac{d}{dt}\Big)^k\Big(\frac{t}{\log(1+t)}\cdot\frac{1}{t}+\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}\frac{D_{m+1}}{m+1}b_{n-m}\Big)\frac{t^n}{n!}\Big).
\end{aligned}\tag{2.34}$$

From (2.33) and (2.34), we have

$$\begin{aligned}
&\Big(\frac{d}{dt}\Big)^k\Big(\frac{\log(1+t)}{t}\cdot\frac{1}{\log(1+t)}\Big)\\
&=\Big(\frac{d}{dt}\Big)^k\left\{\sum_{n=0}^{\infty}\frac{b_{n+1}}{n+1}\cdot\frac{t^n}{n!}+\frac{1}{t}+\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}\frac{D_{m+1}}{m+1}b_{n-m}\Big)\frac{t^n}{n!}\right\}\\
&=\sum_{n=0}^{\infty}\frac{b_{n+1+k}}{n+1+k}\cdot\frac{t^n}{n!}+(-1)^k k!\frac{1}{t^{k+1}}+\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n+k}\binom{n+k}{m}\frac{D_{m+1}}{m+1}b_{n+k-m}\Big)\frac{t^n}{n!}.
\end{aligned}\tag{2.35}$$

By (2.35), we get

$$\begin{aligned}
&(1+t)^k\big(\log(1+t)\big)^{N+1}F^{(k)}\\
&=\Big(\log(1+t)\Big)^{N+1}\left(\sum_{l_1=0}^{\infty}\binom{k}{l_1}t^{l_1}\right)\\
&\times\left\{\sum_{l_2=0}^{\infty}\Big(\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}+\frac{b_{l_2+1+k}}{l_2+1+k}\Big)\frac{t^{l_2}}{l_2!}+(-1)^k k!\frac{1}{t^{k+1}}\right\}\\
&=\Big(\log(1+t)\Big)^{N+1}\\
&\times\left\{\sum_{l=0}^{\infty}\Big(\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}\binom{l}{l_2}+\sum_{l_2=0}^{l}\frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}\binom{l}{l_2}\Big)\frac{t^l}{l!}\right\}\\
&+(-1)^k(N+1)!\sum_{n=N-k}^{\infty}\left\{\sum_{l_3=N-k}^{n}(k)_{n-l_3}\binom{n}{l_3}\frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}}\right\}\frac{t^n}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= (N+1)!\Big(\sum_{l_4=N+1}^{\infty} S_1(l_4,N+1)\frac{t^{l_4}}{l_4!}\Big)\Bigg(\sum_{l=0}^{\infty}\Big(\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}\binom{l}{l_2}\\
&+\sum_{l_2=0}^{l}\frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}\binom{l}{l_2}\Big)\frac{t^l}{l!}\Bigg)\\
&+(-1)^k(N+1)!\sum_{n=N-k}^{\infty}\left\{\sum_{l_3=N-k}^{n}(k)_{n-l_3}\binom{n}{l_3}\frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}}\right\}\frac{t^n}{n!}\\
&= (N+1)!\sum_{n=N+1}^{\infty}\left\{\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}S_1(n-l,N+1)\binom{l}{l_2}\right.\\
&\left.\times\binom{n}{l}\right\}\frac{t^n}{n!}+(N+1)!\sum_{n=N+1}^{\infty}\left(\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}\binom{l}{l_2}\binom{n}{l}\right.\\
&\left.\times S_1(n-l,N+1)\right)\frac{t^n}{n!}+(-1)^k(N+1)!\sum_{n=N-k}^{\infty}\left\{\sum_{l_3=N-k}^{n}(k)_{n-l_3}\binom{n}{l_3}\right.\\
&\left.\times\frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}}\right\}\frac{t^n}{n!}\\
&= (N+1)!\sum_{n=N+1}^{\infty}\left\{\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}S_1(n-l,N+1)\binom{l}{l_2}\right.\\
&\left.\times\binom{n}{l}+\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}\binom{l}{l_2}\binom{n}{l}S_1(n-l,N+1)\right\}\frac{t^n}{n!}\\
&+(-1)^k(N+1)!\sum_{n=N-k}^{\infty}\left\{\sum_{l_3=N-k}^{n}(k)_{n-l_3}\binom{n}{l_3}\frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}}\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{2.36}
$$

From (2.36), we have

$$
\begin{aligned}
&(-1)^N\big(\log(1+t)\big)^{N+1}\sum_{k=0}^{N}a_k(N)(1+t)^kF^{(k)}\\
&= (-1)^N\sum_{k=0}^{N}a_k(N)\\
&\left\{(N+1)!\sum_{n=N+1}^{\infty}\left(\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}S_1(n-l,N+1)\binom{l}{l_2}\right.\right.
\end{aligned}
$$

$$\times \binom{n}{l} + \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} \frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2} S_1(n-l,N+1)\binom{l}{l_2}\binom{n}{l}\Bigg)\frac{t^n}{n!}$$

$$+ (-1)^k (N+1)! \sum_{n=N-k}^{\infty} \left(\sum_{l_3=N-k}^{n} (k)_{n-l_3}\binom{n}{l_3} \frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}} \right)\frac{t^n}{n!}\Bigg\}$$

$$= (N+1)! \sum_{n=N+1}^{\infty} (-1)^N \sum_{k=0}^{N} a_k(N) \Bigg(\sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} \sum_{m=0}^{l_2+k} \frac{D_{m+1}}{m+1} b_{l_2+k-m}(k)_{l-l_2}\binom{l}{l_2}$$

$$\times \binom{n}{l} S_1(n-l,N+1) + \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} \frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2} S_1(n-l,N+1)$$

$$\times \binom{l}{l_2}\binom{n}{l}\Bigg)\frac{t^n}{n!} + (N+1)! \sum_{k=0}^{N} a_k(N)(-1)^{N-k} \sum_{n=N-k}^{\infty} \Bigg(\sum_{l_3=N-k}^{n} (k)_{n-l_3}\binom{n}{l_3}$$

$$\times \frac{S_1(l_3+k+1,N+1)}{(l_3+k+1)\binom{l_3+k}{l_3}}\Bigg)\frac{t^n}{n!}$$

$$= (N+1)! \sum_{n=N+1}^{\infty} \Bigg\{ (-1)^N \sum_{k=0}^{N} \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} \sum_{m=0}^{l_2+k} a_k(N) \frac{D_{m+1}}{m+1} b_{l_2+k-m}(k)_{l-l_2}$$

$$\times S_1(n-l,N+1)\binom{l}{l_2}\binom{n}{l} + (-1)^N \sum_{k=0}^{N} \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} a_k(N) \frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}$$

$$\times \binom{l}{l_2}\binom{n}{l} S_1(n-l,N+1)\Bigg\}\frac{t^n}{n!} + (N+1)! \sum_{k=0}^{N} a_{N-k}(N)(-1)^k$$

$$\times \Bigg\{ \sum_{n=k}^{N} \Bigg(\sum_{l_3=k}^{n} (N-k)_{n-l_3}\binom{n}{l_3} \frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}} \Bigg)\frac{t^n}{n!}$$

$$+ \sum_{n=N+1}^{\infty} \Bigg(\sum_{l_3=k}^{n} (N-k)_{n-l_3}\binom{n}{l_3} \frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}} \Bigg)\frac{t^n}{n!}\Bigg\}$$

$$= (N+1)! \sum_{n=N+1}^{\infty} \Bigg\{ (-1)^N \sum_{k=0}^{N} \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} \sum_{m=0}^{l_2+k} a_k(N) \frac{D_{m+1}}{m+1} b_{l_2+k-m}(k)_{l-l_2}$$

$$\times S_1(n-l,N+1)\binom{l}{l_2}\binom{n}{l} + (-1)^N \sum_{k=0}^{N} \sum_{l=0}^{n-N-1} \sum_{l_2=0}^{l} a_k(N) \frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}$$

$$\times \binom{l}{l_2}\binom{n}{l} S_1(n-l,N+1) + \sum_{k=0}^{N} \sum_{l_3=k}^{n} a_{N-k}(N)(-1)^k (N-k)_{n-l_3}\binom{n}{l_3}$$

$$\times \frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}}\Bigg\}\frac{t^n}{n!}+(N+1)!\sum_{n=0}^{N}\Bigg(\sum_{k=0}^{n}\sum_{l_3=k}^{n}a_{N-k}(N)(-1)^k$$
$$\times (N-k)_{n-l_3}\binom{n}{l_3}\frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}}\Bigg)\frac{t^n}{n!}. \tag{2.37}$$

By (2.1), we set

$$N!\Big(\frac{\log(1+t)}{t}\Big)^{N+1}=N!\sum_{n=0}^{\infty}D_n^{(N+1)}\frac{t^n}{n!}. \tag{2.38}$$

Therefore, by Theorem 1, (2.32) (2.37) and (2.38), we obtain the following theorem.

Theorem 2.3. *Let $N\in\mathbb{N}$,*
(1) For $0\le n\le N$, we have

$$D_n^{(N+1)}=(N+1)\sum_{k=0}^{n}\sum_{l_3=k}^{n}a_{N-k}(N)(-1)^k(N-k)_{n-l_3}\binom{n}{l_3}$$
$$\times\frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}}.$$

(2) For $n\ge N+1$, we have

$$\frac{D_n^{(N+1)}}{N+1}=(-1)^N\sum_{k=0}^{N}\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}\sum_{m=0}^{l_2+k}a_k(N)\frac{D_{m+1}}{m+1}b_{l_2+k-m}(k)_{l-l_2}\binom{l}{l_2}\binom{n}{l}$$
$$\times S_1(n-l,N+1)+(-1)^N\sum_{k=0}^{N}\sum_{l=0}^{n-N-1}\sum_{l_2=0}^{l}a_k(N)\frac{b_{l_2}+1+k}{l_2+1+k}(k)_{l-l_2}\binom{l}{l_2}\binom{n}{l}$$
$$\times S_1(n-l,N+1)+\sum_{k=0}^{N}\sum_{l_3=k}^{n}a_{N-k}(N)(-1)^k(N-k)_{n-l_3}\binom{n}{l_3}$$
$$\times\frac{S_1(l_3+N-k+1,N+1)}{(l_3+N-k+1)\binom{l_3+N-k}{l_3}}.$$

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